2017 年 Yau 赛 (团体赛) 概率部分试题讲解

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1 Let μ^n be the uniform probability measure on the n-dimensional cube $[-1,1]^n$. Let $H \in \mathbb{R}^n$ be the hyperplane orthogonal to the principal diagonal, i.e., $H = (1, \cdots, 1)^{\perp}$. For any $r > 0$, we further define

$$A_{H,r} := \{\boldsymbol{x} \in [-1,1]^n, \operatorname{dist}(\boldsymbol{x}, H) \leqslant r\},$$

where $\operatorname{dist}(\boldsymbol{x}, H)$ represents the distance from the point $\boldsymbol{x}$ to the hyperplane H. Show that for any constant $\varepsilon > 0$, the following two estimates hold for all sufficiently large n

$$(1):\ \mu^n(A_{H,n^\varepsilon}) \geqslant 1 - n^{-2\varepsilon}, \qquad (2):\ \mu^n(A_{H,n^\varepsilon}) \geqslant 1 - \mathrm{e}^{-n^{\varepsilon/2}}.$$

分析. 这题我最开始的想法是尝试计算 $\mu^n(A_{H,n^\varepsilon})$, 即计算 $\dfrac{m(A_{H,n^\varepsilon} \cap [-1,1]^n)}{m([-1,1]^n)}$ (其中 m 为 Lebesgue 测度). 但发现 n 在非常大的时候, 求出 $m(A_{H,n^\varepsilon} \cap [-1,1]^n)$ 的值较为困难. 如果这题在 $n = 2, 3$ 下是完全可以直接求解的, 但是对于一般情形下, 此法难以行通 (需要分类讨论). 进而, 结合题干中的证明目标, 我想到将此问题转化成概率模型, 并利用我们熟知的概率不等式进行求解.

证明. 若 $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in H$, 则有 $x_1 + x_2 + \cdots + x_n = 0$. 结合 $A_{H,r}$ 的定义可知

$$A_{H,n^\varepsilon} := \left\{\boldsymbol{x} \in [-1,1]^n, |x_1 + x_2 + \cdots + x_n| \leqslant n^{\frac{1}{2}+\varepsilon}\right\},$$

我们先将此问题转化成概率模型:
随机变量 $X, X_1, X_2, \cdots, X_n$ i.i.d $U[-1,1]$, 记 $S_n = X_1 + X_2 + \cdots + X_n$, 则 $\mu^n(A_{H,n^\varepsilon}) = \mathbb{P}(|S_n| \leqslant n^{\frac{1}{2}+\varepsilon})$. 故我们只需证明:

$$(1):\ \mathbb{P}(|S_n| > n^{\frac{1}{2}+\varepsilon}) \leqslant n^{-2\varepsilon}, \qquad (2):\ \mathbb{P}(|S_n| > n^{\frac{1}{2}+\varepsilon}) \leqslant \mathrm{e}^{-n^{\varepsilon/2}}.$$

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其中 (1) 通过 Markov 不等式即得:

$$\mathbb{P}(|S_n| > n^{\frac{1}{2}+\varepsilon}) \leqslant \frac{\mathbb{E}(|S_n|^2)}{n^{1+2\varepsilon}} = \frac{\sum_{k=1}^n \mathbb{E}(X_k^2)}{n^{1+2\varepsilon}} = \frac{\mathbb{E}(X^2)}{n^{2\varepsilon}} \leqslant n^{-2\varepsilon}.$$

对于 (2), 我们先证明 Hoeffding 不等式:

引理 (Hoeffding 不等式). 设随机变量 $X_1, X_2, \cdots, X_n$ 独立, 且满足 $a_i \leqslant X_i \leqslant b_i$. 记 $S_n = X_1 + X_2 + \cdots + X_n$, 则对 $\forall x > 0$, 有

$$\mathbb{P}(S_n - \mathbb{E}S_n \geqslant nx) \leqslant \exp\left\{-\frac{2n^2x^2}{\sum_{i=1}^n (b_i - a_i)^2}\right\}.$$

证明. 设 $\mathbb{E}X_i = 0$, 对任意 $t > 0$, 有

$$\mathbb{P}(S_n \geqslant nx) \leqslant \mathrm{e}^{-tnx}\mathbb{E}\mathrm{e}^{tS_n} = \mathrm{e}^{-tnx}\prod_{k=1}^n \mathbb{E}\mathrm{e}^{tX_k}. \tag{1}$$

下面我们估计 $\mathbb{E}\mathrm{e}^{tX_i}$. 取 $\gamma = \dfrac{x - a_i}{b_i - a_i}$, 利用 Jenson 不等式, 我们有

$$\mathrm{e}^{tx} := f(x) = f(\gamma a_i + (1-\gamma)b_i) \leqslant \gamma f(a_i) + (1-\gamma) f(b_i) \leqslant \frac{x - a_i}{b_i - a_i}\mathrm{e}^{tb_i} + \frac{b_i - x}{b_i - a_i}\mathrm{e}^{ta_i}.$$

因此

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{tX_i} &\leqslant -\frac{a_i}{b_i - a_i}\mathrm{e}^{tb_i} + \frac{b_i}{b_i - a_i}\mathrm{e}^{ta_i} \\
&= (1 - \theta + \theta \mathrm{e}^{t(b_i - a_i)})\mathrm{e}^{-\theta t(b_i - a_i)} \\
&= (1 - \theta + \theta \mathrm{e}^{u})\mathrm{e}^{-\theta u} \\
&= \mathrm{e}^{g(u)}.
\end{aligned}$$

其中

$$\theta = -\frac{a_i}{b_i - a_i},\ u = t(b_i - a_i),\ g(u) = -\theta u + \log(1 - \theta + \theta \mathrm{e}^u).$$

而 $g(0) = g'(0) = 0,\ g''(u) \leqslant \dfrac{1}{4}(\forall u > 0)$. 利用 Taylor 展开, 存在 $\xi \in (0, u)$, 满足

$$g(u) = g(0) + g'(0)u + \frac{g''(\xi)}{2}u^2 \leqslant \frac{u^2}{8} = \frac{t^2(b_i - a_i)^2}{8}.$$

因此 $\mathbb{E}\mathrm{e}^{tX_i} \leqslant \mathrm{e}^{\frac{t^2(b_i - a_i)^2}{8}}$. 将上式代入 (1) 得

$$\mathbb{P}(S_n \geqslant nx) \leqslant \exp\left\{-tnx + \frac{t^2}{8}\sum_{i=1}^n (b_i - a_i)^2\right\}$$

取 $t = \dfrac{4nx}{\sum_{i=1}^n (b_i - a_i)^2}$ 即得该不等式. □

回到原题, 利用 $\mathbb{E}S_n=0$ 及 Hoeffding 不等式, 我们有

$$\mathbb{P}(|S_n|>n^{\frac{1}{2}+\varepsilon})=2\mathbb{P}(\frac{S_n}{n}>n^{\varepsilon-\frac{1}{2}})\leqslant 2\mathrm{e}^{-\frac{1}{2}n^{2\varepsilon}}$$

当 n 充分大时, 有

$$\mathbb{P}(|S_n|>n^{\frac{1}{2}+\varepsilon})\leqslant 2\mathrm{e}^{-\frac{1}{2}n^{2\varepsilon}}\leqslant \mathrm{e}^{-n^{\varepsilon/2}}.$$

成立. □

注. 1. (2) 中的证明中用到了 Hoeffding 不等式, 这个不等式比较强. 如果直接通过矩母函数估计做法更简单, 利用 $\dfrac{\mathrm{e}^t-\mathrm{e}^{-t}}{2t}\leqslant \mathrm{e}^{\frac{t^2}{2}}$(请读者自证) 可以得到

$$\mathbb{P}(S_n\geqslant nx)\leqslant \mathrm{e}^{-tnx}\mathbb{E}\mathrm{e}^{tS_n}=\mathrm{e}^{-tnx}\prod_{k=1}^{n}\mathbb{E}\mathrm{e}^{tX_k}=\mathrm{e}^{-tnx}\left(\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{2t}\right)^n\leqslant \exp\left\{-tnx+\frac{1}{2}nt^2\right\}.$$

这里 $x=n^{\varepsilon-\frac{1}{2}}$. 取 $t=x^2$, 则有

$$\mathbb{P}(|S_n|>n^{\frac{1}{2}+\varepsilon})\leqslant 2\mathbb{P}(S_n>n^{\frac{1}{2}+\varepsilon})\leqslant 2\mathrm{e}^{-\frac{1}{2}n^{2\varepsilon}}.$$

2. 有关概率模型的转化可以思考下面两个问题:
(1) 设 U,V 是独立地均匀地取自 n 维单位超立方体上的两点, X_n 表示两点欧氏距离, 证明:

$$\frac{\mathbb{E}X_n}{\sqrt{n}}\to\frac{1}{\sqrt{6}}.$$

(2) 设向量 $\boldsymbol{v_1},\boldsymbol{v_2},\cdots,\boldsymbol{v_n}\in\mathbb{R}^n$, 且对任意 $1\leqslant i\leqslant n$ 满足 $|\boldsymbol{v_i}|\leqslant 1$. 对 $\boldsymbol{w}=\sum\limits_{i=1}^{n}p_i\boldsymbol{v_i}$, 其中 $p_i\in[0,1]$. 证明: 存在 $\varepsilon_i\in\{0,1\}$, 使得

$$\left|\sum_{i=1}^{n}\varepsilon_i\boldsymbol{v_i}-\boldsymbol{w}\right|\leqslant\frac{\sqrt{n}}{2}.$$

2 Let $X_1,X_2,\cdots$ be positive random variables. We assume that X_n converges to 0 in probability, and that $\lim\limits_{n\to\infty}\mathbb{E}X_n=2$. Prove that $\lim\limits_{n\to\infty}\mathbb{E}|X_n-1|$ exists and compute its value.

解. 直接计算即可. 我们有

$$\begin{aligned}\mathbb{E}|X_n-1|&=\mathbb{E}((X_n-1)\mathbb{1}_{\{X_n\geqslant 1\}})+\mathbb{E}((1-X_n)\mathbb{1}_{\{X_n<1\}})\\&=\mathbb{E}(X_n\mathbb{1}_{\{X_n\geqslant 1\}})-\mathbb{E}(X_n\mathbb{1}_{\{X_n<1\}})-\mathbb{P}(X_n\geqslant 1)+\mathbb{P}(X_n<1)\\&=\mathbb{E}X_n+1-2\mathbb{E}(X_n\mathbb{1}_{\{X_n<1\}})-2\mathbb{P}(X_n\geqslant 1).\end{aligned}$$

由 $X_n \xrightarrow{P} 0$ 知 $\lim_{n\to\infty}\mathbb{P}(X_n \geqslant 1) = 0$. 又因为 $X_n\mathbb{1}_{\{X_n<1\}}$ 有界, 由控制收敛定理知 $\lim_{n\to\infty}\mathbb{E}(X_n\mathbb{1}_{\{X_n<1\}}) = 0$. 故有

$$\lim_{n\to\infty}\mathbb{E}|X_n - 1| = 3.$$

□

3 There are n people playing a game. Initially everybody had one dollar at hand. During each round of the game, we randomly pick two people and they will toss a fair coin, to decide who wins this round of the game. The loser will submit one dollar (note: just one, not all of his money) to the winner. Assume that a person who had no money at hand will be immediately driven out of the game. The game stops until all money is at the hand of only one person. Calculate the average number of rounds that the game plays.

Note: In each round only two players are involved.

分析. 开始我先想到直接通过递推的方法解决, 但是这样做会发现需要讨论的情形总数非常之多, 且与 n 呈正相关, 很难入手. 后来我就把研究对象放在一个人身上, 发现其本质就回到常见的双吸收壁模型, 这样就找到了本题的切入点. 通过局部的分析最后再回到整体上即可.

解. 我们先证明如下有关双吸收壁模型的引理 (详见 **Durrett 4.8 节**):

Let $\xi_1, \xi_2, \ldots$ be i.i.d., $S_n = S_0 + \xi_1 + \cdots + \xi_n$, where S_0 is a constant, and let $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. We will now derive some result by using the three martingales from Section 4.2.

Theorem 4.8.7 (Symmetric simple random walk) *refers to the special case in which $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Suppose $S_0 = x$ and let $N = \min\{n : S_n \notin (a,b)\}$. Writing a subscript x to remind us of the starting point*

$$(a) \qquad P_x(S_N = a) = \frac{b-x}{b-a} \qquad P_x(S_N = b) = \frac{x-a}{b-a}$$

(b) $E_0 N = -ab$ and hence $E_x N = (b-x)(x-a)$.

Let $T_x = \min\{n : S_n = x\}$. Taking $a = 0$, $x = 1$ and $b = M$ we have

$$P_1(T_M < T_0) = \frac{1}{M} \qquad P_1(T_M < T_0) = \frac{M-1}{M}$$

The first result proves (4.4.1). Letting $M \to \infty$ in the second we have $P_1(T_0 < \infty) = 1$.

Proof (a) To see that $P(N < \infty) = 1$ note that if we have $(b-a)$ consecutive steps of size $+1$, we will exit the interval. From this it follows that

$$P(N > m(b-a)) \leq (1 - 2^{-(b-a)})^m$$

so $EN < \infty$.

Clearly, $E|S_N| < \infty$ and $S_n 1_{\{N>n\}}$ are uniformly integrable, so using Theorem 4.8.2, we have

$$x = ES_N = aP_x(S_N = a) + b[1 - P_x(S_N = a)]$$

Rearranging, we have $P_x(S_N = a) = (b-x)/(b-a)$ subtracting this from 1, $P_x(S_N = b) = (x-a)/(b-a)$.
(b) The second result is an immediate consequence of the first.
Using the stopping theorem for the bounded stopping time $N \wedge n$, we have

$$0 = E_0 S_{N\wedge n}^2 - E_0(N \wedge n)$$

The monotone convergence theorem implies that $E_0(N \wedge N) \uparrow E_0 N$. Using the bounded convergence theorem and the result of (a) with $x = 0$ implies

$$\begin{aligned} E_0 S_{N\wedge n}^2 &\to a^2 \frac{b}{b-a} + b^2 \frac{-a}{b-a} \\ &= -ab\left[\frac{-a}{b-a} + \frac{b}{b-a}\right] = -ab \end{aligned}$$

which completes the proof. □

以上证明需要用到鞅与停时定理, 下面给出递推方法的证明:

证明. 同上述证明, 我们有 $\mathbb{P}(N < \infty) = 1$. 故 $\mathbb{P}_x(S_N = a) + \mathbb{P}_x(S_N = b) = 1$. 记 $p_x = \mathbb{P}_x(S_N = a)$, 则有 $p_a = 1, p_b = 0$. 利用全概率公式, 有

$$p_x = \frac{1}{2}(p_{x+1} + p_{x-1}).$$

化简得

$$p_{x+1} - p_x = p_x - p_{x-1}.$$

再把 $p_a = 1, p_b = 0$ 代入, 得

$$\mathbb{P}_x(S_N = a) = p_x = \frac{b-x}{b-a}, \quad \mathbb{P}_x(S_N = b) = \frac{x-a}{b-a}.$$

同理我们可以求 $\mathbb{E}_x N$. 设 $T_x = \mathbb{E}_x N$, 则有 $T_a = T_b = 0$. 取条件有

$$T_x = \frac{1}{2}(T_{x+1} + 1) + \frac{1}{2}(T_{x-1} + 1) = \frac{1}{2}(T_{x+1} + T_{x-1}) + 1.$$

化简得

$$T_{x+1} - T_x = T_x - T_{x-1} - 2$$

再利用 $0 = T_b - T_a = \sum_{k=a}^{b-1}(T_{k+1} - T_k)$, 可得

$$T_{x+1} - T_x = b + a - 2x - 1.$$

所以

$$\mathbb{E}_x N = T_x = T_x - T_a = \sum_{k=a}^{x-1}(T_{k+1} - T_k) = \sum_{k=a}^{x-1}(b + a - 2k - 1) = (b-x)(x-a).$$

□

回到原题. 记 T 为整场游戏结束时的总轮数, $X_i(1 \leqslant i \leqslant n)$ 为第 i 个人在到达 0 dollar 或 n dollars 时他已投掷过硬币的总次数, 我们发现, 最后游戏结束时所有人均处于 0 dollar 或 n dollars 的状态, 且到达这两个状态的人均已结束游戏, 之后不再投掷硬币. 而每一轮游戏中有且仅有两个人发生投掷硬币的事件, 据此列出关系式

$$2T = X_1 + X_2 + \cdots + X_n.$$

由引理知, 对任意 $1 \leqslant i \leqslant n$, 均有 $\mathbb{E}X_i = n-1$, 因此

$$\mathbb{E}T = \frac{1}{2}\mathbb{E}(X_1 + X_2 + \cdots + X_n) = \frac{n}{2}\mathbb{E}X_1 = \frac{n(n-1)}{2}.$$

□

注. 这个解法很好地体现了数学中“局部与整体”的思想。事实上, 我们在数学上接触到的“局部与整体”更多是在概念中体现, 但在数学模型问题中也有关于“局部与整体”的处理方法，这里就是一个很好的例子.

4 Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{X'_n\}_{n\in\mathbb{N}}$ be two independent simple random walks on $\mathbb{Z}^d$ such that $X_0 = X'_0 = 0$. Here simple walk means if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^d$ and $||\boldsymbol{x}-\boldsymbol{y}|| = 1$, then

$$\mathbb{P}(X_{n+1} = \boldsymbol{y} | X_n = \boldsymbol{x}) = (2d)^{-1}.$$

Let $I = \{(s,t) : X_s = X'_t\}$. Prove that $|I| < \infty$ *a.s.*

证明. 我们首先证明:

$$\mathbb{P}(X_n = 0) = O(n^{-\frac{d}{2}}) \quad (n \to \infty).$$

我们有

$$\begin{aligned}
\mathbb{P}(X_{2n+1} = 0) &= 0, \\
\mathbb{P}(X_{2n} = 0) &= \sum_{\substack{i_1,\cdots,i_{d-1}\geqslant 0 \\ i_1+\cdots+i_{d-1}\leqslant n}} \frac{(2n)!}{(i_1!)^2(i_2!)^2\cdots(i_{d-1}!)^2((n-i_1-\cdots-i_{d-1})!)^2}(2d)^{-2n} \\
&= 2^{-2n}\binom{2n}{n} \sum_{\substack{i_1,\cdots,i_{d-1}\geqslant 0 \\ i_1+\cdots+i_{d-1}\leqslant n}} \left(d^{-n}\frac{n!}{i_1!i_2!\cdots i_{d-1}!(n-i_1-\cdots-i_{d-1})!}\right)^2 \\
&\leqslant 2^{-2n}\binom{2n}{n} \max_{\substack{i_1,\cdots,i_{d-1}\geqslant 0 \\ i_1+\cdots+i_{d-1}\leqslant n}} \left(d^{-n}\frac{n!}{i_1!i_2!\cdots i_{d-1}!(n-i_1-\cdots-i_{d-1})!}\right)
\end{aligned}$$

$$\times \sum_{\substack{i_1,\cdots,i_{d-1}\geqslant 0\\ i_1+\cdots+i_{d-1}\leqslant n}} d^{-n}\frac{n!}{i_1!i_2!\cdots i_{d-1}!(n-i_1-\cdots-i_{d-1})!}$$

一方面,

$$\sum_{\substack{i_1,\cdots,i_{d-1}\geqslant 0\\ i_1+\cdots+i_{d-1}\leqslant n}} \frac{n!}{i_1!i_2!\cdots i_{d-1}!(n-i_1-\cdots-i_{d-1})!}=(\underbrace{1+1+\cdots+1}_{d\text{个}})^n=d^n.$$

另一方面, 存在 $C_1>0$, 使得

$$\max_{\substack{i_1,\cdots,i_{d-1}\geqslant 0\\ i_1+\cdots+i_{d-1}\leqslant n}}\left(d^{-n}\frac{n!}{i_1!i_2!\cdots i_{d-1}!(n-i_1-\cdots-i_{d-1})!}\right)\leqslant C_1 d^{-n}\frac{n!}{\left(\frac{n}{d}!\right)^d}$$

利用 Stirling 公式 $n!\sim\sqrt{2n\pi}\left(\frac{n}{\mathrm{e}}\right)^n\ (n\to\infty)$, 我们有

$$C_1 d^{-n}\frac{n!}{\left(\frac{n}{d}!\right)^d}\xrightarrow{n\to\infty} d^{-n}\frac{n^n}{(\left(\frac{n}{d}\right)^{n/d})^d}\sqrt{\frac{n}{\left(\frac{n}{d}\right)^d}}\frac{1}{(2\pi)^{\frac{d-1}{2}}}\leqslant C_2 n^{-\frac{d-1}{2}},$$

其中 C_2 为大于 0 的常数. 再由 Stirling 公式知,

$$\mathbb{P}(X_{2n}=0)=C_2 2^{-2n}\binom{2n}{n}n^{-\frac{d-1}{2}}\leqslant C_3 n^{-\frac{d}{2}},$$

其中 C_3 为大于 0 的常数. 所以

$$\mathbb{P}(X_n=0)=O(n^{-\frac{d}{2}})\quad (n\to\infty).$$

回到原题, 当 $d=1,2$ 时, $\mathbb{P}(X_n=0)\sim n^{-\frac{d}{2}}$ (计算过程和上述类似), 而 $\sum\limits_{n=1}^{\infty}n^{-\frac{d}{2}}=+\infty$, 故有 $\sum\limits_{n=1}^{\infty}\mathbb{P}(X_n=0)=+\infty$. 因此 X_n 在 0 处常返. 而对 $\forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}^d$, $\boldsymbol{x},\boldsymbol{y}$ 均相互可达, 因此对 $\forall\boldsymbol{x}\in\mathbb{Z}^d$, X_n 在 $\boldsymbol{x}$ 处常返. 记 $\tau_{\boldsymbol{x}}=\inf\{k\geqslant 1|X_k=\boldsymbol{x}\}$, $\tau'_{\boldsymbol{x}}=\inf\{k\geqslant 1|X'_k=\boldsymbol{x}\}$. 由常返性知 $\mathbb{P}(\tau_{\boldsymbol{x}}<\infty)=\mathbb{P}(\tau'_{\boldsymbol{x}}<\infty)=1$. 则对任意 $N\in\mathbb{N}^*$, 均有

$$\mathbb{P}(I\geqslant N)\geqslant\mathbb{P}(\cap_{i=1}^N(\{\tau_{\boldsymbol{x}_i}<\infty\}\cap\{\tau_{\boldsymbol{x}_i}<\infty\})\geqslant 1-\sum_{i=1}^N(\mathbb{P}(\tau_{\boldsymbol{x}}=\infty)+\mathbb{P}(\tau'_{\boldsymbol{x}}=\infty))=1.$$

即

$$\mathbb{P}(I\geqslant N)=1.$$

所以 $|I|=\infty$ $a.s.$. 故 $d=1,2$ 时原命题不成立.

现考虑 $d \geqslant 5$ 时的情形. 我们只需证明 $\mathbb{E}I < \infty$(否则若 $\mathbb{P}(I = \infty) > 0$, 则有 $\mathbb{E}I = \infty$, 矛盾). 记 $p_{\boldsymbol{ij}}(n) = \mathbb{P}(X_n = \boldsymbol{j}|X_0 = \boldsymbol{i}), \rho_{\boldsymbol{ij}} = \mathbb{P}(\exists n \in N^*, s.t. X_n = \boldsymbol{j}|X_0 = \boldsymbol{i})$. 利用一个强马氏链的结论: $\sum_{k=0}^{n} p_{\boldsymbol{ij}}(k) = \rho_{\boldsymbol{ij}} \sum_{k=0}^{n} p_{\boldsymbol{jj}}(k)$ (单调收敛定理保证 $n = \infty$ 结论亦成立), 我们有

$$\begin{aligned}
\mathbb{E}I &= \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \mathbb{E}(\mathbb{1}_{\{X_s = X'_t\}}) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\boldsymbol{j}\in\mathbb{Z}^d} \mathbb{P}(X_s = \boldsymbol{j})\mathbb{P}(X'_t = \boldsymbol{j}) \\
&= \sum_{\boldsymbol{j}\in\mathbb{Z}^d}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \mathbb{P}(X_s = \boldsymbol{j})\mathbb{P}(X'_t = \boldsymbol{j}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d}\left(\sum_{s=0}^{\infty}\mathbb{P}(X_s = \boldsymbol{j})\right)\left(\sum_{t=0}^{\infty}\mathbb{P}(X'_t = \boldsymbol{j})\right) \\
&= \sum_{\boldsymbol{j}\in\mathbb{Z}^d}\left(\sum_{s=0}^{\infty} p_{0\boldsymbol{j}}(s)\right)\left(\sum_{t=0}^{\infty} p_{0\boldsymbol{j}}(t)\right) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d}\rho_{0\boldsymbol{j}}^2\left(\sum_{s=0}^{\infty} p_{\boldsymbol{jj}}(s)\right)\left(\sum_{t=0}^{\infty} p_{\boldsymbol{jj}}(t)\right) \\
&= \sum_{\boldsymbol{j}\in\mathbb{Z}^d}\rho_{0\boldsymbol{j}}^2\left(\sum_{s=0}^{\infty} p_{00}(s)\right)\left(\sum_{t=0}^{\infty} p_{00}(t)\right) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d}\rho_{0\boldsymbol{j}}^2\left(\sum_{s=0}^{\infty} p_{00}(s)\right)^2,
\end{aligned}$$

这里单调收敛定理保证求和次序的可交换性. 由 $\mathbb{P}(X_n = 0) = O(n^{-\frac{d}{2}})$ 知,

$$\sum_{s=0}^{\infty} p_{00}(s) < +\infty.$$

记 $||X|| = \sup_{1\leqslant i\leqslant d} |x_i|$ $(X = (x_1, \cdots, x_d))$, 利用群里文献《SOME INTERSECTION PROPERTIES OF RANDOM WALK PATHS》中的 Lemma1 可知, 存在 $C > 0$, 对任意 $k \in \mathbb{N}^*$ 及 $||\boldsymbol{j}|| = k$, 有

$$\rho_{0\boldsymbol{j}} < \frac{C}{k^{d-2}}.$$

因此, 存在 $C_0 > 0$, 有

$$\begin{aligned}
\sum_{\boldsymbol{j}\in\mathbb{Z}^d}\rho_{0\boldsymbol{j}}^2 &\leqslant 1 + \sum_{\substack{k=1\\ ||\boldsymbol{j}||=k}}^{\infty}\rho_{0\boldsymbol{j}}^2 \leqslant 1 + \sum_{\substack{k=1\\ ||\boldsymbol{j}||=k}}^{\infty}\frac{C^2}{k^{2d-4}} = 1 + \sum_{k=1}^{\infty}(k^d - (k-1)^d)\cdot\frac{C^2}{k^{2d-4}} \\
&\leqslant 1 + C_0\sum_{k=1}^{\infty}\frac{k^{d-1}}{k^{2d-4}} = 1 + C_0\sum_{k=1}^{\infty}k^{3-d} < +\infty.
\end{aligned}$$

故 $\mathbb{E}I < \infty$ $a.s.$ 成立.

另解: 关于 $d \geqslant 5$ 的情形, 谢晟捷同学给出了一个更加简单的方法:

$$\begin{aligned}
\mathbb{E}I &= \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\mathbb{E}(\mathbb{1}_{\{X_s = X'_t\}}) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\mathbb{P}(X_s = X'_t) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\mathbb{P}(X_s + X'_t = 0) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\mathbb{P}(X_{s+t} = 0) \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{n}\mathbb{P}(X_n = 0) = \sum_{n=0}^{\infty}(n+1)\mathbb{P}(X_n = 0) \leqslant C'\sum_{n=0}^{\infty}n^{1-\frac{d}{2}} < +\infty.
\end{aligned}$$

关于 $d = 3, 4$ 的情形:

请大家参考文献《SOME INTERSECTION PROPERTIES OF RANDOM WALK PATHS》中的证明, 结果是 $\mathbb{E}I = \infty$ $a.s.$, 即原命题不成立. 希望强大的群友们能想出更好的方法, , , $\square$